

SOURCE INTEGRALS FOR MULTIPOLE MOMENTS IN STATIC SPACETIMES

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We derive source integrals for multipole moments that describe the behavior of static and axially symmetric spacetimes close to spatial infinity. We assume that the matter distribution is isolated. We outline also some applications of these source integrals of the asymptotic multipole moments.

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1. Introduction

The multipole moments of a Newtonian mass distribution can be determined in different ways. First, we can expand the gravitational potential far away from the source. The coefficients of the resulting series are then identified with the multipole moments. Secondly, the moments of the mass density μ , i.e. integrals of products of μ and certain polynomials, equal the multipole moments. For instance, we have for axially symmetric distributions and the multipole moments M_i in polar coordinates (r, θ, φ) :

$$U = - \sum_{i=0}^{\infty} \frac{M_i}{r^{i+1}} P_i(\cos \theta), \quad M_i = \int_V \mu r^i P_i(\cos \theta) dV,$$

where P_i denote the Legendre polynomials of the first kind and V is the support of μ . Geometric units are chosen such that $G = c = 1$. That these two definitions are equivalent is not trivial but can be proved using the Poisson integral. In general relativity, both types of definitions exist but they are in general not equivalent. The first type of definition involves only the gravitational field close to space-like or null-like infinity and it yields *asymptotic multipole moments* (or field multipole moments¹). This approach was taken in Refs. 2–11, for reviews see Ref. 10,12. The second type of definitions characterizes solely the source and will be called *source multipole moments* or *source integrals*. Such definitions were put forward, e.g. in Ref. 1,13.

In this paper, we show that both types of definitions can be related to each other also in general relativity, i.e., we give a source integral formulation of asymptotic multipole moments. Details of the calculations can be found in Refs. 14,15. Such a formalism has many applications, e.g. in the description of tidal deformations of neutron stars and black holes. It also allows to characterize exact and numerical solutions by their asymptotic multipole moments using only the metric in the interior of the matter. We give an example of that in the end of this paper.

2. Formalism

We concentrate on axially symmetric and static spacetimes. In Weyl form they read:

$$ds^2 = e^{2k-2U} (d\rho^2 + d\zeta^2) + W^2 e^{-2U} d\varphi^2 - e^{2U} dt^2.$$

The metric functions U , W and k depend only on the Weyl coordinates ρ and ζ . With the time-like and hypersurface orthogonal Killing vector ξ^a and the space-like Killing vector η^a , the metric functions are given by

$$e^{2U} = -\xi_a \xi^a, \quad W^2 = -\xi_a \xi^a \eta_b \eta^b.$$

Furthermore, the Killing vectors define a 1-form

$$Z_a = \epsilon_{abcd} W^{,b} W^{-1} \eta^c \xi^d$$

that is hypersurface orthogonal everywhere as well as exact in the vacuum region. Hence, a potential Z and an integrating factor X exist such that $Z_{,a} = X Z_a$, where $X = 1$ in the exterior of a topological 2-sphere \mathcal{S}_0 , which encloses all sources. In the vacuum region and in *canonical* Weyl coordinates, we have $Z = \zeta + \text{const.}$ and $W = \rho$. Since we can shift the ζ -coordinate freely, we can drop the constant of integration. It specifies the origin with respect to which the asymptotic multipole moments are introduced. Using the canonical Weyl coordinates, we can define Weyl's asymptotic multipole moments $U^{(r)}$ covariantly via an expansion of U along the axis of symmetry $\rho = 0$ close to spatial infinity:

$$U(\rho = 0, \zeta) = \sum_{r=0}^{\infty} U^{(r)} |\zeta|^{-r-1}.$$

The $U^{(r)}$ are sufficient to calculate the Geroch-Hansen multipole moments.¹⁶

Let us further introduce two sets of polynomials in W and Z :

$$N_r^- = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \frac{2(-1)^{k+1} r! W^{2k+1} Z^{r-2k}}{4^k (k!)^2 (r-2k)!}, \quad N_r^+ = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \frac{2(-1)^{k+1} r! W^{2k+2} Z^{r-2k-1}}{4^k (k!)^2 (r-2k-1)! (2k+2)}.$$

$\lfloor x \rfloor$ denotes the greatest integer y with $y \leq x$. Moreover, suppose that each black hole is enclosed by a topological 2-sphere \mathcal{S}_i that contains no other sources and assume that there is a (not necessarily connected) region \mathcal{V} in the projection orthogonal to ξ^a that covers all matter. Then the source integrals evaluate to^{14,15}

$$\begin{aligned} U_r &= \frac{1}{8\pi} \int_{\mathcal{V}} \rho_r d\mathcal{V} + \frac{1}{8\pi} \sum_i \int_{\mathcal{S}_i} \frac{e^U}{W} \left(N_r^- U_{,\hat{n}} - N_{r,W}^+ Z_{,\hat{n}} U + N_{r,Z}^+ W_{,\hat{n}} U \right) d\mathcal{S}_i, \\ \rho_r &= e^U \left[-\frac{N_r^-}{W} R_{ab} \frac{\xi^a \xi^b}{\xi^c \xi_c} + N_{r,Z}^+ U \left(\frac{W^{,a}}{W} \right)_{;a} - N_{r,W}^+ U \left(\frac{Z^{,a}}{W} \right)_{;a} + \right. \\ &\quad \left. N_{r,WZ}^+ \frac{U}{W} (W^{,a} W_{,a} - Z^{,a} Z_{,a}) \right], \end{aligned} \quad (1)$$

where $d\mathcal{S}_i$ and $d\mathcal{V}$ denote the proper surface or volume element of \mathcal{S}_i and \mathcal{V} , respectively. $f_{,\hat{n}}$ is the derivative of f in the direction of the unit normal of \mathcal{S}_i .

We conclude the paper discussing one possible application of the source integrals (1). Assume a matter distribution is given and the metric is known in \mathcal{V} . Even then it is far from trivial (at least in the stationary case¹⁷) to obtain a global asymptotically flat solution, if it exists. The source integrals provide a tool to solve this task. We demonstrated the procedure for static dust configurations in Refs. 14,15. There we showed that all asymptotic multipole moments of static and axially symmetric dust configurations can be calculated and that they all vanish. This contradicts the presence of a dust distribution with a positive mass density. Of course, this result is already known and more general non-existence results for dust including the rotating case can be found in Refs. 18,19 and references therein. Although the non-existence was proved,^{14,15} this example shows in a concise way how the source integrals can be applied in more difficult physical situations like rotating relativistic stars. This and other applications, e.g. to tidal distortions of black holes, will be investigated in future work.

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